lsjk – a C++ library for arbitrary-precision numeric evaluation of the generalized log-sine functions

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Abstract

Generalized log-sine functions $\mathrm{Ls}_j^{(k)}(\theta)$ appear in higher order ε-expansion of different Feynman diagrams. We present an algorithm for the numerical evaluation of these functions for real arguments. This algorithm is implemented as a C++ library with arbitrary-precision arithmetics for integer $0 \leq k \leq 9$ and $j \geq 2$. Some new relations and representations of the generalized log-sine functions are given.

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PROGRAM SUMMARY

Title of program: lsjk

Version: 1.0.0 *Release*: 1.0.0 *Catalogue number*:

Program obtained from: `http://thsun1.jinr.ru/~varg/dist/`

E-mail: varg@thsun1.jinr.ru

Licensing terms: GNU General Public Licence

Computers: all

Operating systems: POSIX

Programming language: `C++`

Memory required to execute: Depending on the complexity of the problem, at least 32Mb RAM recommended.

Other programs called: The CLN library for arbitrary-precision arithmetics is required at version 1.1.5 or greater.

External files needed: none

Keywords: Generalized log-sine functions, Feynman integrals, Polylogarithms.

Nature of the physical problem: Numerical evaluation of the generalized log-sine functions for real argument in the region $0 < \theta < \pi$. These functions appear in Feynman integrals.

Method of solution: Series representation for the real argument in the region $0 < \theta < \pi$.

Restriction on the complexity of the problem: Limited up to $\mathrm{Ls}_j^{(9)}(\theta)$, and j is an arbitrary integer number. Thus, all function up to the weight 12 in the region $0 < \theta < \pi$ can be evaluated. The algorithm can be extended up to higher values of k ($k > 9$) without modification.

Typical running time: Depending on the complexity of problem. See Table 1 in section 3.4

LONG WRITE-UP

1 Introduction

Quite recently it has been realized that generalized log-sine functions [1] play an important role in analytical calculations of multiloop Feynman diagrams. Within dimensional regularization [2] with an arbitrary space-time dimension $d = 4 - 2\varepsilon$, the log-sine functions $\mathrm{Ls}_j(\theta)$ appear in the construction of the *all* order ε-expansion of the one-loop propagator type diagrams with arbitrary masses [3], three-point integrals with massless internal lines and arbitrary (off-shell) external momenta [4], two-loop vacuum diagrams with arbitrary masses [4, 5] and two-loop single scale diagrams of propagator type [6]. The appearance of the generalized log-sine functions $\mathrm{Ls}_j^{(k)}(\theta)$ $(k = 1, 2)$ was first detected in the finite part of the three-loop bubble master-integrals [7, 8]. More examples of the one-, two- and three-loop Feynman diagrams whose ε-expansion contains the generalized log-sine functions can be found in [8–13]. The generalized log-sine functions are also related to the derivatives of the generalized hypergeometric functions with respect to their parameters [13, 14] and to the multiple Euler–Zagier sums [7, 8]. A collection of the known analytical properties of these functions can be found in Appendix A of [8] (see also [15]).

Here we present C++ code for arbitrary precision numeric evaluation of the generalized function $\mathrm{Ls}_j^{(k)}(\theta)$ of the real argument $0 < \theta < \pi$ and $k \le 9$. For calculation of the $\mathrm{Ls}_j(\theta)$ and $\mathrm{Ls}_j^{(1)}(\theta)$ functions there exists a FORTRAN program (see the description in [16]).

The plan of the paper is the following. Section 2 contains the definition and some properties of the generalized log-sine functions. Section 3 is the description of the lsjk library. In Appendix A we present relations between the functions $\mathrm{Ls}_j^{(0,1,2)}(\theta)$ and infinite series containing the trigonometric functions and harmonic sums.

2 Generalized log-sine functions

2.1 Definition and reflection symmetries

The generalized log-sine functions are defined as [1]

$$\mathrm{Ls}_j^{(k)}(\theta) = -\int_0^\theta \mathrm{d}\phi\ \phi^k \ln^{j-k-1}\left|2\sin\frac{\phi}{2}\right| , \quad \mathrm{Ls}_j(\theta) = \mathrm{Ls}_j^{(0)}(\theta) \ , \tag{2.1}$$

where k, j are integer numbers, $k \ge 0$ and $j \ge k+1$ and θ is an arbitrary real number.

According to the definition (2.1), the following integral and differential relations hold:

$$\frac{1}{\alpha^k}\mathrm{Ls}_{j+k}^{(i+k)}(\alpha\theta) = \theta^k \mathrm{Ls}_j^{(i)}(\alpha\theta) - k\int_0^\theta \mathrm{d}\phi\ \phi^{k-1}\mathrm{Ls}_j^{(i)}(\alpha\phi) \ , \tag{2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Ls}_{j+k}^{(i+k)}(\alpha\theta) = (\alpha\theta)^k \frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Ls}_j^{(i)}(\alpha\theta) \ , \tag{2.3}$$

where α is a real number. In particular,

$$\mathrm{Ls}_j^{(j-1)}(\theta) = -\frac{1}{j}\theta^j \ .$$

It is easy to obtain a relation between functions of the opposite arguments

$$\mathrm{Ls}_j^{(k)}(-\theta) = (-1)^{k+1}\mathrm{Ls}_j^{(k)}(\theta) \ , \quad \theta > 0 \ . \tag{2.4}$$

Another relation was deduced in Ref. [1] via the integral representation:

$$\mathrm{Ls}_n^{(r)}(2m\pi-\theta) = \mathrm{Ls}_n^{(r)}(2m\pi)+(-1)^{r-1}\mathrm{Ls}_n^{(r)}(\theta)-\sum_{p=1}^{r}(-1)^{r-p}\,(2m\pi)^p \binom{r}{p}\mathrm{Ls}_{n-p}^{(r-p)}(\theta) \ , \tag{2.5}$$

where m is an integer number. Using the symmetry property (2.4), this relation can be written as

$$\mathrm{Ls}_n^{(r)}(2m\pi+\theta) = \mathrm{Ls}_n^{(r)}(2m\pi)+\mathrm{Ls}_n^{(r)}(\theta)+\sum_{p=1}^{r}(-1)^{r-p}\,(2m\pi)^p \binom{r}{p}\mathrm{Ls}_{n-p}^{(r-p)}(\theta) \ . \tag{2.6}$$

The generalized log-sine functions appear in the decomposition of polylogarithms $\mathrm{Li}_j\left(1-e^{\mathrm{i}\theta}\right)$ into real and imaginary parts (see Eqs. (A.7) in [8]). It is easy to show that the generalized Nielsen polylogarithms [17] $S_{a,b}(z)$ of a unit-circle complex argument $z = e^{\mathrm{i}\theta}$ also reduce to a combination of the generalized log-sine integrals (2.1) [1]

$$\begin{aligned}
S_{a,b}\left(e^{\mathrm{i}\theta}\right) &= \frac{\mathrm{i}^a(-1)^b}{(a-1)!\,b!}\int_0^\theta \mathrm{d}\phi\;(\theta-\phi)^{a-1}\left[\ln\left|2\sin\tfrac{\phi}{2}\right|-\tfrac{1}{2}\mathrm{i}(\pi-\phi)\right]^b+\sum_{k=0}^{a-1}\frac{(\mathrm{i}\theta)^k}{k!}S_{a-k,b}(1) \\
&= \frac{(-\mathrm{i})^a(-1)^b}{(a-1)!\,b!}\sum_{k=0}^{a-1}\sum_{j=0}^{b}\sum_{m=0}^{j}(-\theta)^{a-1-k}(-\pi)^{j-m}\left(\frac{\mathrm{i}}{2}\right)^j\binom{a-1}{k}\binom{b}{j}\binom{j}{m}\mathrm{Ls}_{m+k+b+1-j}^{(m+k)}(\theta) \\
&+\sum_{k=0}^{a-1}\frac{(\mathrm{i}\theta)^k}{k!}S_{a-k,b}(1) \ .
\end{aligned} \tag{2.7}$$

2.2 Identities between the generalized log-sine and Clausen functions

It was shown in [8] that the function $\mathrm{Ls}_{k+2}^{(k)}(\theta)$ is always expressible in terms of the Clausen functions $\mathrm{Cl}_j(\theta)$. This follows from the fact that $\mathrm{Ls}_2(\theta) = \mathrm{Cl}_2(\theta)$, the differential identity (2.3), which in this case has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Ls}_{k+2}^{(k)}(\theta) = \theta^k\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Cl}_2(\theta) \ , \tag{2.8}$$

[1] The corresponding equation in Ref. [10] contains an misprint in the last term. We are grateful to A. Kotikov for correspondence.

and the integration rules for the Clausen function [1],

$$\mathrm{Cl}_{2n}(\theta) = \int\limits_0^\theta \mathrm{d}\phi\, \mathrm{Cl}_{2n-1}(\phi)\,, \quad \mathrm{Cl}_{2n+1}(\theta) = \zeta_{2n+1} - \int\limits_0^\theta \mathrm{d}\phi\, \mathrm{Cl}_{2n}(\phi)\,.$$

The general solution of Eq. (2.8) could be represented in the following form

$$\mathrm{Ls}_{2j+2}^{(2j)}(\theta) = \sum_{k=0}^{2j} \frac{(2j)!}{(2j-k)!} \theta^{2j-k} \mathrm{Cl}_{2+k}(\theta)\, (-1)^{k(k-1)/2}\,, \tag{2.9}$$

$$\begin{aligned} \mathrm{Ls}_{2j+3}^{(2j+1)}(\theta) &= (-1)^j (2j+1)!\, [\mathrm{Cl}_{2j+3}(\theta) - \zeta_{2j+3}] \\ &\quad + \sum_{k=0}^{2j} \frac{(2j+1)!}{(2j+1-k)!} \theta^{2j+1-k} \mathrm{Cl}_{2+k}(\theta)\, (-1)^{k(k-1)/2}\,. \end{aligned} \tag{2.10}$$

The corresponding expressions up to $\mathrm{Ls}_6^{(4)}(\theta)$ have been presented in [8].

2.3 Identities between the generalized log-sine functions and the generalized Nielsen polylogarithms

In Ref. [8] the following relations between the log-sine functions and generalized Nielsen polylogarithms were deduced:

$$\begin{aligned} \mathrm{i}\sigma\, [\mathrm{Ls}_j(\pi) - \mathrm{Ls}_j(\theta)] &= \frac{1}{2^j j} \ln^j(-z) \left[1 - (-1)^j\right] \\ &+ (-1)^j (j-1)! \sum_{p=0}^{j-2} \frac{\ln^p(-z)}{2^p p!} \sum_{k=1}^{j-1-p} (-2)^{-k} \left[S_{k,j-k-p}(z) - (-1)^p S_{k,j-k-p}(1/z)\right], \end{aligned} \tag{2.11}$$

where the relation between z and θ is defined via

$$z \equiv e^{\mathrm{i}\sigma\theta}, \quad \ln(-z - \mathrm{i}\sigma 0) = \ln(z) - \mathrm{i}\sigma\pi, \quad \sigma = \pm 1, \tag{2.12}$$

and $0 \leq \theta \leq \pi$. For higher values of j the number of generalized polylogarithms involved in Eq. (2.11) can be reduced (see Eqs. (2.21), (2.22) in [8]). The relation between $\mathrm{Ls}_{j+1}^{(1)}(\theta)$ and the generalized Nielsen polylogarithms is also known (see Eq. (A.20) in [8]),

$$\begin{aligned} \mathrm{Ls}_{j+1}^{(1)}(\theta) - \mathrm{Ls}_{j+1}^{(1)}(\pi) &= \theta \Big[\mathrm{Ls}_j(\theta) - \mathrm{Ls}_j(\pi)\Big] - \frac{1}{2^j j(j+1)} \ln^{j+1}(-z) \left[1 - (-1)^j\right] \\ &- (-1)^j (j-1)! \sum_{p=0}^{j-2} \frac{\ln^p(-z)}{2^p p!} \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} k \left[S_{k+1,j-k-p}(z) + (-1)^p S_{k+1,j-k-p}(1/z)\right] \\ &+ 2(-1)^j (j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(-1)\,. \end{aligned} \tag{2.13}$$

Using the algorithm described in [8] we found the following relation for $\mathrm{Ls}_{j+2}^{(2)}(\theta)$ and generalized Nielsen polylogarithms:

$$
\begin{aligned}
& \mathrm{i}\sigma\left[\mathrm{Ls}_{j+2}^{(2)}(\pi)-\mathrm{Ls}_{j+2}^{(2)}(\theta)\right]=2\mathrm{i}\sigma\theta\left[\mathrm{Ls}_{j+1}^{(1)}(\pi)-\mathrm{Ls}_{j+1}^{(1)}(\theta)\right]-\mathrm{i}\sigma\theta^{2}\left[\mathrm{Ls}_{j}(\pi)-\mathrm{Ls}_{j}(\theta)\right] \\
& -\frac{1}{2^{j-1}j(j+1)(j+2)}\ln^{j+2}(-z)\left[1-(-1)^{j}\right] \\
& -(-1)^{j}(j-1)!\sum_{p=0}^{j-2}\frac{\ln^{p}(-z)}{2^{p}p!}\sum_{k=1}^{j-1-p}\frac{(-1)^{k}k(k+1)}{2^{k}}\left[S_{k+2,j-k-p}(z)-(-1)^{p}S_{k+2,j-k-p}(1/z)\right] \\
& +4(-1)^{j}(j-1)!\ln(-z)\sum_{k=1}^{j-1}\frac{(-1)^{k}}{2^{k}}kS_{k+1,j-k}(-1)\;. \qquad (2.14)
\end{aligned}
$$

The procedure, described in [8], can be repeated several times, so that the generalized log-sine functions with arbitrary indices can be directly related to Nielsen polylogarithms.

2.4 Series representation

In the region $0 < \theta < \pi$, the series representation of the generalized log-sine functions directly follows from the definition (2.1) by substituting $y = \sin\frac{\phi}{2}$

$$
\mathrm{Ls}_{j}^{(k)}(\theta) = -2^{k+1}\int_{0}^{\sin(\theta/2)}\frac{\mathrm{d}y}{\sqrt{1-y^2}}\left(\arcsin y\right)^{k}\ln^{j-k-1}(2y). \qquad (2.15)
$$

Using integration by parts, this integral can be rewritten in the following form:

$$
\mathrm{Ls}_{j}^{(k)}(\theta) = -\frac{\theta^{k+1}}{k+1}\ln^{j-k-1}\left(2\sin\frac{\theta}{2}\right)+\frac{2^{k+1}}{k+1}(j-k-1)\int_{0}^{\sin(\theta/2)}\frac{\mathrm{d}y}{y}\left(\arcsin y\right)^{k+1}\ln^{j-k-2}(2y). \qquad (2.16)
$$

Expanding $\left(\arcsin y\right)^{k}$ in y (see below) and using

$$
\int x^{a}\ln^{b}x\;\mathrm{d}x = (-1)^{b}\;b!\;x^{a+1}\sum_{p=0}^{b}\frac{\left(-\ln x\right)^{b-p}}{(b-p)!(a+1)^{p+1}}\;,
$$

we obtain the multiple series representation. The generating function for the coefficients of the Taylor expansion of $\left(\arcsin y\right)^{k}$ and their explicit form for $k = 1, 2, 3, 4$ can be found in [18]. We have calculated the coefficients of the Taylor expansion for $k = 5, \cdots, 12$. For even powers, the Taylor expansion of $\left(\arcsin y\right)^{2k}$ can be represented in the following form:

$$
\frac{\left(\arcsin y\right)^{2k}}{(2k)!} = \sum_{m=1}^{\infty}\frac{[(m-1)!]^{2}}{(2m)!}4^{m-1}y^{2m}B(m,k)\;, \qquad (2.17)
$$

where the coefficients $B(m,k)$ are expressible in terms of harmonic sums

$$
S_{a}(n) = \sum_{k=1}^{n}\frac{1}{k^{a}}\;. \qquad (2.18)
$$

For the lowest values of k we have

$$\begin{aligned}
&B(m,1)=1\ , \quad B(m,2)=\sigma_1\ , \quad B(m,3)=\frac{1}{2}\ \left[\sigma_1^2-\sigma_2\right]\ , \\
&B(m,4)=\frac{1}{3!}\ \left[\sigma_1^3-3\sigma_1\sigma_2+2\sigma_3\right]\ , \\
&B(m,5)=\frac{1}{4!}\ \left[\sigma_1^4+8\sigma_1\sigma_3+3\sigma_2^2-6\sigma_4-6\sigma_1^2\sigma_2\right]\ , \\
&B(m,6)=\frac{1}{5!}\ \left[\sigma_1^5-30\sigma_1\sigma_4-20\sigma_2\sigma_3+20\sigma_1^2\sigma_3-10\sigma_1^3\sigma_2+15\sigma_1\sigma_2^2+24\sigma_5\right]\ ,
\end{aligned} \tag{2.19}$$

and

$$\sigma_a=\sum_{j=1}^{m-1}\frac{1}{(2j)^{2a}}\equiv\frac{1}{4^a}S_{2a}(m-1)\ . \tag{2.20}$$

For odd powers, the Taylor expansion of $(\arcsin y)^{2k+1}$ has the following structure:

$$\frac{(\arcsin y)^{2k+1}}{(2k+1)!}=\sum_{m=0}^{\infty}\binom{2m}{m}\frac{y^{2m+1}}{4^m(2m+1)}C(m,k+1)\ , \tag{2.21}$$

where the coefficients $C(m,k)$ also expressible in terms of harmonic sums,

$$\begin{aligned}
&C(m,1)=1\ , \quad C(m,2)=h_1\ , \quad C(m,3)=\frac{1}{2}\ \left[h_1^2-h_2\right]\ , \\
&C(m,4)=\frac{1}{3!}\ \left[h_1^3-3h_1h_2+2h_3\right]\ , \\
&C(m,5)=\frac{1}{4!}\ \left[h_1^4+8h_1h_3+3h_2^2-6h_4-6h_1^2h_2\right]\ , \\
&C(m,6)=\frac{1}{5!}\ \left[h_1^5-30h_1h_4-20h_2h_3+20h_1^2h_3-10h_1^3h_2+15h_1h_2^2+24h_5\right]\ ,
\end{aligned} \tag{2.22}$$

and

$$h_a=\sum_{j=1}^{m}\frac{1}{(2j-1)^{2a}}\equiv S_{2a}(2m-1)-\frac{1}{4^a}S_{2a}(m-1)\ . \tag{2.23}$$

Starting with Eq. (2.16) and using expansions (2.17) and (2.21), we get

$$\begin{aligned}
\mathrm{Ls}_j^{(2k)}(\theta)=(-1)^j\left(2\sin\tfrac{\theta}{2}\right)2^{2k}(2k)!(j-2k-1)!\sum_{p=0}^{j-2k-1}\frac{\left[-\ln\left(2\sin\frac{\theta}{2}\right)\right]^p}{p!}& \\
\times\sum_{m=0}^{\infty}\binom{2m}{m}\frac{\left(2\sin\frac{\theta}{2}\right)^{2m}}{16^m(2m+1)^{j-2k-p}}C(m,k+1)\ ,&
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
\mathrm{Ls}_j^{(2k+1)}(\theta)=(-1)^{j-1}2^{2k+2}(2k+1)!(j-2k-2)!\sum_{p=0}^{j-2k-2}\frac{\left[-\ln\left(2\sin\frac{\theta}{2}\right)\right]^p}{p!}& \\
\times\sum_{m=1}^{\infty}\frac{1}{\binom{2m}{m}}\frac{\left(2\sin\frac{\theta}{2}\right)^{2m}}{(2m)^{j-2k-p}}B(m,k+1)\ ,&
\end{aligned} \tag{2.25}$$

where $k = 0, 1, 2, 3, 4, 5$ and the coefficients $B(m,k)$ and $C(m,k)$ are given in (2.19) and (2.22), respectively.

The special interest in the calculation of the multiloop Feynman diagrams [7, 8, 10, 19] is related to the value of the generalized log-sine functions at the "sixth root of unity", $\theta = \frac{\pi}{3}$. In this case $\ln\left(2\sin\frac{\theta}{2}\right) = 0$ and only the $p = 0$ term in the finite sum in Eqs. (2.24),(2.25) survives,

$$\mathrm{Ls}_j^{(2k)}\left(\tfrac{\pi}{3}\right) = (-1)^j 2^{2k}(2k)!(j-2k-1)! \sum_{m=0}^{\infty} \binom{2m}{m} \frac{C(m,k+1)}{16^m(2m+1)^{j-2k}} , \tag{2.26}$$

$$\mathrm{Ls}_j^{(2k+1)}\left(\tfrac{\pi}{3}\right) = (-1)^{j-1} 2^{2k+2}(2k+1)!(j-2k-2)! \sum_{m=1}^{\infty} \frac{1}{\binom{2m}{m}} \frac{B(m,k+1)}{(2m)^{j-2k}} . \tag{2.27}$$

Similar series for $\mathrm{Ls}_j(\theta)$ and $\mathrm{Ls}_j^{(1)}(\theta)$ have been presented earlier in [16].

2.5 Special values of the argument

The values of the generalized log-sine functions of the argument $\theta = \pi$ and 2π can be related with the values of some well-known special functions. The relations of this type were discussed in Lewin's book [1]. For completeness we present some of them below. The values of $\mathrm{Ls}_j(\pi)$ can be expressed in terms of the Rieman ζ-function, for any j (see Eqs. (7.108), (7.110) in [1]),

$$\mathrm{Ls}_{n+1}(\pi) = -\pi \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left.\frac{\Gamma(1+x)}{\Gamma^2(1+\frac{x}{2})}\right|_{x=0} = -\pi \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp\left[\sum_{m=2}^{\infty} \frac{x^m}{m}(-1)^m \left(1-2^{1-m}\right)\zeta_m\right]_{x=0} \tag{2.28}$$

so that

$$\mathrm{Ls}_2(\pi) = 0,\;\; \mathrm{Ls}_3(\pi) = -\tfrac{1}{2}\pi\zeta_2,\;\; \mathrm{Ls}_4(\pi) = \tfrac{3}{2}\pi\zeta_3,\;\; \mathrm{Ls}_5(\pi) = -\tfrac{57}{8}\pi\zeta_4,\;\; \mathrm{Ls}_6(\pi) = \tfrac{45}{2}\pi\zeta_5 + \tfrac{15}{2}\pi\zeta_2\zeta_3,$$

etc. Putting in Eq. (2.5) $r = 0$ and $\theta = 2\pi(m-1)$, solving the difference equation and taking into account the relation between values of function $\mathrm{Ls}_j(\theta)$ at $\theta = \pi$ and 2π, we finally get

$$\mathrm{Ls}_n(2m\pi) \;=\; 2m\mathrm{Ls}_n(\pi) . \tag{2.29}$$

From relation (2.5) for $r = 1$ and $\theta = 2\pi m$ we have a new relation

$$\mathrm{Ls}_n^{(1)}(2\pi m) \;=\; 2m^2\pi\mathrm{Ls}_{n-1}(\pi) . \tag{2.30}$$

Putting $r = 3$ and $\theta = 2\pi m$ in Eq. (2.5) and using Eqs. (2.29) (2.30) we get

$$\mathrm{Ls}_n^{(3)}(2\pi m) \;=\; 3\pi m\mathrm{Ls}_{n-1}^{(2)}(2\pi m) - 4m^4\pi^3\mathrm{Ls}_{n-3}(\pi) , \tag{2.31}$$

where the value of $\mathrm{Ls}_m^{(2)}(2\pi m)$ should be calculated independently. The integral $\mathrm{Ls}_n^{(m)}(2\pi)$ can be expressed in terms of a combination of ζ-functions (see Eq. (7.136) in [1]):

$$\mathrm{Ls}_{n+m+1}^{(n)}(2\pi) \;=\; -2\pi(-\mathrm{i})^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^m e^{\mathrm{i}x\pi} \left.\frac{\Gamma(1+y)}{\Gamma(1+\frac{y}{2}+x)\Gamma(1+\frac{y}{2}-x)}\right|_{x=y=0} . \tag{2.32}$$

It can be written in the following form [2]:

$$\begin{aligned}
\mathrm{Ls}_{n+m+1}^{(n)}(2\pi) &= -2\pi(-\mathrm{i})^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^m \\
&\times \exp\left[\mathrm{i}x\pi + \sum_{p=2}^{\infty} \frac{y^p}{p!}(-1)^p \left(1-2^{1-p}\right)\zeta_p \right.\\
&\left. - \sum_{q=2}^{\infty} \frac{x^q}{q!}[1+(-1)^q]\left\{(q-1)!\zeta_q + \sum_{p=1}^{\infty} \frac{y^p}{p!}\left(\frac{-1}{2}\right)^p \Gamma(p+q)\zeta_{p+q}\right\}\right]_{x=y=0}, \quad (2.33)
\end{aligned}$$

so that for $n=2$ we get [3]

$$\begin{aligned}
\mathrm{Ls}_{m+3}^{(2)}(2\pi) &= \frac{8}{3}\pi^2 \mathrm{Ls}_{m+1}(\pi) - \pi(m+1)!(-2)^{2-m}\zeta_{2+m} \\
&+ 4m! \sum_{p=1}^{m-1} \frac{(-2)^{p-m}}{p!}(m-p+1)\mathrm{Ls}_{p+1}(\pi)\,\zeta_{2+m-p}, \quad m \ge 2. \quad (2.34)
\end{aligned}$$

In particular, we have [4]

$$\mathrm{Ls}_5^{(2)}(2\pi) = -\frac{13}{45}\pi^5, \quad \mathrm{Ls}_6^{(2)}(2\pi) = 5\pi^3\zeta_3 + 12\pi\zeta_5, \quad \mathrm{Ls}_7^{(2)}(2\pi) = -\frac{29}{105}\pi^7 - 24\pi\zeta_3^2. \quad (2.35)$$

Substituting expression (2.34) in Eq. (2.31) we get ($m=1$)

$$\begin{aligned}
\mathrm{Ls}_n^{(3)}(2\pi) &= 4\pi^3 \mathrm{Ls}_{n-3}(\pi) - 3\pi^2(n-3)!(-2)^{6-n}\zeta_{n-2} \\
&+ 3\pi(n-4)! \sum_{p=1}^{n-5} \frac{(-2)^{p-n+6}}{p!}(n-p-3)\mathrm{Ls}_{p+1}(\pi)\,\zeta_{n-2-p}, \quad n \ge 5, \quad (2.36)
\end{aligned}$$

and

$$\mathrm{Ls}_5^{(3)}(2\pi) = 12\pi^2\zeta_3, \quad \mathrm{Ls}_6^{(3)}(2\pi) = -\frac{8}{15}\pi^6, \quad \mathrm{Ls}_7^{(3)}(2\pi) = 9\pi^4\zeta_3 + 36\pi^2\zeta_5. \quad (2.37)$$

The values of the generalized log-sine functions of the argument π are related to the values of the Γ-functions and generalized Nielsen polylogarithms of the argument $z=-1$ (see section 7.9.9 in [1]). For lower values of k ($k=0,1,2$) the proper relations can be deduced from Eqs. (2.11) – (2.14):

$$\begin{aligned}
\mathrm{Ls}_j(\pi) &= \frac{\mathrm{i}^{j+1}}{2^j j}\pi^j \left[1-(-1)^j\right] \\
&+ (-1)^j(j-1)! \sum_{p=0}^{j-2} \frac{\mathrm{i}^{p+1}}{p!}\left(\frac{\pi}{2}\right)^p \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} S_{k,j-k-p}(1)\left[1-(-1)^p\right], \quad (2.38)
\end{aligned}$$

[2] The term $-[1+(-1)^n](n-1)!\zeta_n$ is absent in Eq. (7.139) of [1].

[3] In Eq. (7.141) of [1], the term $2\pi^2\mathrm{Ls}_{r+1}(\pi)$ should read $\frac{8}{3}\pi^2\mathrm{Ls}_{r+1}(\pi)$.

[4] The correct result for $\mathrm{Ls}_5^{(2)}(2\pi)$ was first given in [8].

$$\mathrm{Ls}_{j+1}^{(1)}(\pi) = \frac{\mathrm{i}^{j+1}}{2^j j(j+1)} \pi^{j+1} \left[1-(-1)^j\right] - 2(-1)^j (j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(-1) \; .$$

$$+(-1)^j (j-1)! \sum_{p=0}^{j-2} \frac{(\mathrm{i}\pi)^p}{2^p p!} \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} k S_{k+1,j-k-p}(1) \left[1+(-1)^p\right] \; , \tag{2.39}$$

$$\mathrm{Ls}_{j+2}^{(2)}(\pi) = -\frac{\mathrm{i}^{j+3}}{2^{j-1} j(j+1)(j+2)} \pi^{j+2} \left[1-(-1)^j\right] - 4\pi (-1)^j (j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(-1)$$

$$-(-1)^j (j-1)! \sum_{p=0}^{j-2} \frac{\mathrm{i}^{p+1}}{p!} \left(\frac{\pi}{2}\right)^p \sum_{k=1}^{j-1-p} \frac{(-1)^k k(k+1)}{2^k} S_{k+2,j-k-p}(1) \left[1-(-1)^p\right] \; . \tag{2.40}$$

In particular, we have

$$\mathrm{Ls}_4^{(1)}(\pi) = -2S_{2,2}(-1) - \tfrac{11}{8}\zeta_4 \; , \tag{2.41}$$

$$\mathrm{Ls}_5^{(1)}(\pi) = 6S_{2,3}(-1) + \tfrac{3}{2}\zeta_2\zeta_3 - \tfrac{105}{32}\zeta_5 \; , \quad \mathrm{Ls}_5^{(2)}(\pi) = 2\pi \mathrm{Ls}_4^{(1)}(\pi) + 2\pi\zeta_4 \; , \tag{2.42}$$

where the values of generalized Nielsen polylogarithms of argument $z = \pm 1$ can be extracted from [17, 20]. For completeness we present the proper results below

$$\begin{aligned}
S_{a,b}(1) &= S_{b,a}(1) \; , \quad S_{2,2}(1) = \tfrac{1}{4}\zeta_4 \; , \quad S_{1,3}(1) = \zeta_4 \; , \quad S_{2,3}(1) = 2\zeta_5 - \zeta_2\zeta_3 \; , \\
S_{1,3}(-1) &= \tfrac{1}{24}\ln^4 2 - \tfrac{1}{4}\zeta_2 \ln^2 2 + \tfrac{7}{8}\zeta_3 \ln 2 - \zeta_4 + \mathrm{Li}_4\left(\tfrac{1}{2}\right) \; , \\
S_{2,3}(-1) &= -\tfrac{1}{15}\ln^5 2 + \tfrac{1}{3}\zeta_2 \ln^3 2 - \tfrac{7}{8}\zeta_3 \ln^2 2 + \tfrac{1}{2}\zeta_2\zeta_3 + \tfrac{33}{32}\zeta_5 - 2\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln 2 - 2\mathrm{Li}_5\left(\tfrac{1}{2}\right) \; . \\
S_{2,2}(-1) &= 2S_{1,3}(-1) + \tfrac{1}{8}\zeta_4 \; , \quad S_{3,2}(-1) = \tfrac{1}{2}\zeta_3\zeta_2 - \tfrac{29}{32}\zeta_5 \; .
\end{aligned} \tag{2.43}$$

3 The lsjk library

3.1 Domain

Our algorithm for the numerical evaluation of the generalized log-sine functions in the region $0 < \theta < \pi$ is based on the multiple series representation (2.24) and (2.25). The explicit value of the generalized log-sine function at $\theta = \pi$ and $\theta = 2\pi$ can be extracted by the approach described in Lewin's book [1] (see also section 2.5). For an argument belonging to the region $\pi < \theta < 2\pi$ the following representation is valid

$$\mathrm{Ls}_j^{(k)}(\theta) = (-1)^{k+1} \sum_{p=0}^{k} (-2\pi)^p \binom{k}{p} \left\{ \mathrm{Ls}_{j-p}^{(k-p)}(2\pi - \theta) - \mathrm{Ls}_{j-p}^{(k-p)}(2\pi) \right\} . \tag{3.1}$$

For negative values of argument, $\theta < 0$, the reflection symmetry (2.4) is applied. Eqs. (2.5) and (2.6) allow us to express the value of the generalized log-sine function of the argument $|\theta| > 2\pi$ in terms of the function $\mathrm{Ls}_k^{(j)}(\theta)$ at $0 < \theta < \pi$, π and 2π.

In the present version of lsjk we have considered only the region $0 < \theta < \pi$.

3.2 Description

For arbitrary-precision arithmetics, lsjk uses the CLN library [5]. CLN has a rich set of number classes:

- Integers (with unlimited precision)
- Rational numbers (with unlimited precision)
- Floating-point numbers:
  - Short float
  - Single float
  - Double float
  - Long float numbers (with unlimited precision)
- Complex numbers

and implements many elementary functions on these numbers.

lsjk provides two functions [6]

```
const cl_R Ls(const unsigned & j, const cl_R &x);
const cl_R Ls(const unsigned & j, const unsigned & k, const cl_R &x);
```

These functions represent $\mathrm{Ls}_j(x)$ and $\mathrm{Ls}_j^{(k)}(x)$, respectively. The argument x is supposed to satisfy $0 < x < \pi$, parameter k should be $0 \le k \le 9$ [7]. lsjk can be used as any regular library, e.g. by writing programs and linking executables to it.

3.3 Installation instructions

The installation procedure is reduced to the simple three steps[8] of

```
./configure && make && make install
```

The "configure" script can be given a number of options to enable and disable various features. A few of the more important ones are documented in the "INSTALL" file from the lsjk distribution.

After installation lsjk can be used as any regular library, e.g. by writing programs and linking executables with it. lsjk includes shell script "lsjk-config" that can be used for setting compiler and linker command-line options required to compile and link a program with the lsjk library. Thus, a simple program can be compiled as

```
c++ -o simple ‘lsjk-config --cppflags‘ simple.cpp ‘lsjk-config --libs‘
```

See "README" file from the lsjk distribution for more details.

[5] CLN can be downloaded from http://www.ginac.de/CLN. Alternatively, you may wish to check your favored operating system distribution for a precompiled package.

[6] cl_R is CLN type for real numbers, see CLN manual for details.

[7] In the present version we have implemented only first five coefficients $B(m,k)$ and $C(m,k)$, $k = 1,2,3,4,5$ from Eqs. (2.19) and (2.22), respectively.

[8] last step may require superuser privileges

	$j=4$			$j=5$				$j=6$			
$d \setminus k$	0	1	2	0	1	2	3	0	1	2	3
128	0.053	0.034	0.055	0.055	0.037	0.058	0.048	0.059	0.04	0.063	0.052
256	0.166	0.106	0.187	0.172	0.111	0.195	0.16	0.177	0.118	0.203	0.166
512	0.685	0.448	0.872	0.699	0.47	0.876	0.742	0.727	0.491	0.9	0.741
1024	3.212	2.219	5.084	3.294	2.292	4.459	4.013	3.327	2.342	4.496	3.786

Table 1: Dependence of the average calculation time (in second), as reported by getrusage(2), of $\mathrm{Ls}_j^{(k)}(2\pi/3)$ on the precision (d) of evaluation. It was measured on a Duron/800MHz with 128Mb RAM.

3.4 Benchmarks and cross-checks

Correctness of implementation was cross-checked by comparison with results of numerical integration [9] of Eq. (2.1) for $4 \leq j \leq 11$, $0 \leq k \leq 9$, with randomly chosen argument $0 < x < \pi$. We also checked that the results of the evaluation of the functions $\mathrm{Ls}_j(\theta)$ and $\mathrm{Ls}_j^{(1)}(\theta)$ coincide with the corresponding results of the FORTRAN program described in [16].

To give a reader a rough idea about the evaluation time, Table 1 may be helpful. It should be noted that the calculation time for values of the argument near π increases substantially.

3.5 Example of a program

The following program evaluates $\mathrm{Ls}_5^{(2)}(2\pi/3)$ up to 1024 digits.

```
#include <iostream>
#include <cln/cln.h>
#include <lsjk/lsjk.hpp>
using namespace std;
using namespace cln;

int main(int argc, char** argv)
{
        const unsigned j=5;
        const unsigned k=2;
        float_format_t prec=float_format(1024); // precision
        cl_R ThePi = pi(prec);
        cl_R res = Ls(j, k, 2*ThePi/3);
        cout << res << endl;
        return 0;
}
```

The output of this program is:

[9]For numerical integration we used GNU scientific library, see http://www.gnu.org/software/gsl

```
-0.518108786829680117347265638731696755021879668243153214067389472482 4\
```

4 Conclusion

In this paper we have described the C++ library lsjk for arbitrary-precision numerical evaluation of the generalized log-sine functions for the real argument $0 < \theta < \pi$. The evaluation is based on the series representations (2.24) and (2.25). The present version allows one to evaluate generalized log-sine functions up to $\mathrm{Ls}_j^{(9)}(\theta)$, thus, all functions up to weight 11 can be evaluated. The algorithm can be extended to higher weights without modification. In particular, implementation of the next coefficients $B(m, 6)$ and $C(m, 6)$ (see Eqs. (2.19) and (2.22)) allows one to evaluate all functions up to weight 14.

Explicit formulae (2.14) relating $\mathrm{Ls}_j^{(2)}(\theta)$ and the generalized Nielsen polylogarithms have been obtained. Using the relations of this type for the functions $\mathrm{Ls}_j(\theta)$ and $\mathrm{Ls}_j^{(1)}(\theta)$, the relations (A.2)–(A.4) between series included trigonometric functions and harmonic sums and the functions $\mathrm{Ls}_j^{(0,1,2)}(\theta)$ have been deduced. As a particular case, some new identities (A.12)-(A.17) for a special type of harmonic sums have been obtained. Another application of these expressions is the relation between the values of functions $\mathrm{Ls}_j^{(1,2)}(\theta)$ at $\theta = \pi$ and the Nielsen polylogarithms at $z = \pm 1$ (2.39), (2.40). The correct expressions for the functions $\mathrm{Ls}_j^{(2,3)}(\theta)$ at $\theta = 2\pi$ in terms of the Γ-functions and their derivatives are obtained (2.34), (2.36).

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A Identities between $\mathrm{Ls}_j^{(0,1,2)}(\theta)$ and infinite series

Identities between the generalized log-sine functions and infinite series involving trigonometric functions and harmonic sums follow from the power series expansion of the generalized Nielsen polylogarithms

$$S_{n,p}(z) = \sum_{k=1}^{\infty} \begin{bmatrix} k \\ p \end{bmatrix} \frac{z^k}{k!k^n} \; , \tag{A.1}$$

where $\left[\begin{smallmatrix} k \\ p \end{smallmatrix}\right]$ denotes Stirling numbers of first kind defined as [21]

$$\ln^p(1-z) = (-1)^p p! \sum_{k=p}^{\infty} \begin{bmatrix} k \\ p \end{bmatrix} \frac{z^k}{k!} \; .$$

Using Eqs. (2.11) and (A.1) we get

$$\begin{aligned} \mathrm{Ls}_j(\theta) - \mathrm{Ls}_j(\pi) = {} & \frac{1}{2^j j}(-1)^{j-1}\mathrm{i}^{j-1}(\pi-\theta)^j \left[1-(-1)^j\right] \\ & -(-1)^j(j-1)! \sum_{p=0}^{j-2} \frac{(-\mathrm{i}\sigma)^{p+1}(\pi-\theta)^p}{2^p p!} \sum_{k=1}^{j-1-p} (-2)^{-k} \sum_{n=1}^{\infty} \begin{bmatrix} n \\ j-k-p \end{bmatrix} \frac{1}{n!n^k} \\ & \times \left\{\cos(n\theta)\left[1-(-1)^p\right] + \mathrm{i}\sigma\sin(n\theta)\left[1+(-1)^p\right]\right\} . \end{aligned} \tag{A.2}$$

Similar expansion can be deduced for $\mathrm{Ls}_j^{(1)}(\theta)$ and $\mathrm{Ls}_j^{(2)}(\theta)$ with the help of Eqs. (2.13) and (2.14),

$$\begin{aligned} & \mathrm{Ls}_{j+1}^{(1)}(\theta) - \mathrm{Ls}_{j+1}^{(1)}(\pi) = \theta\Big[\mathrm{Ls}_j(\theta) - \mathrm{Ls}_j(\pi)\Big] - \frac{\mathrm{i}^{j+1}}{2^j j(j+1)}(\pi-\theta)^{j+1}\left[1-(-1)^j\right] \\ & -(-1)^j(j-1)! \sum_{p=0}^{j-2} \frac{(-\mathrm{i}\sigma)^p(\pi-\theta)^p}{2^p p!} \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} k \sum_{n=1}^{\infty} \begin{bmatrix} n \\ j-k-p \end{bmatrix} \frac{1}{n!n^{k+1}} \\ & \times \left\{\cos(n\theta)\left[1+(-1)^p\right] + \mathrm{i}\sigma\sin(n\theta)\left[1-(-1)^p\right]\right\} \\ & +2(-1)^j(j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(-1) \; , \end{aligned} \tag{A.3}$$

$$\begin{aligned} & \mathrm{Ls}_{j+2}^{(2)}(\theta) - \mathrm{Ls}_{j+2}^{(2)}(\pi) = 2\theta\Big[\mathrm{Ls}_{j+1}^{(1)}(\theta) - \mathrm{Ls}_{j+1}^{(1)}(\pi)\Big] - \theta^2\Big[\mathrm{Ls}_j(\theta) - \mathrm{Ls}_j(\pi)\Big] \\ & + \frac{\mathrm{i}^{j+3}}{2^{j-1} j(j+1)(j+2)}(\pi-\theta)^{j+2}\left[1-(-1)^j\right] \\ & +(-1)^j(j-1)! \sum_{p=0}^{j-2} \frac{(-\mathrm{i}\sigma)^{p+1}(\pi-\theta)^p}{2^p p!} \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} k(k+1) \sum_{n=1}^{\infty} \begin{bmatrix} n \\ j-k-p \end{bmatrix} \frac{1}{n!n^{k+2}} \\ & \times \left\{\cos(n\theta)\left[1-(-1)^p\right] + \mathrm{i}\sigma\sin(n\theta)\left[1+(-1)^p\right]\right\} \\ & +4(\pi-\theta)(-1)^j(j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(-1) \; , \end{aligned} \tag{A.4}$$

For lower values of j these expressions have the following form:

$$\mathrm{Ls}_3(\theta) = -2\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}S_1 - \tfrac{1}{12}\theta\left(\theta^2-3\pi\theta+3\pi^2\right)\ , \tag{A.5}$$

$$\mathrm{Ls}_4(\theta) = 3\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}\left[S_1^2-S_2+\tfrac{1}{4}(\pi-\theta)^2-\tfrac{1}{2}\frac{1}{n^2}\right]-\tfrac{3}{2}(\pi-\theta)\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^3}+\tfrac{3}{2}\pi\zeta_3\ , \tag{A.6}$$

$$\mathrm{Ls}_5(\theta) = 3(\pi-\theta)\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}\left[S_1^2-S_2\right]+3\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^3}\left[S_1^2-S_2\right]$$
$$-4\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}\left[S_1^3-3S_1S_2+2S_3\right]-\tfrac{1}{16}\pi\theta(\pi-\theta)(\pi^2-\pi\theta+\theta^2)-\tfrac{1}{80}\theta^5-\tfrac{1}{15}\pi^5\ , \tag{A.7}$$

$$\mathrm{Ls}_4^{(1)}(\theta) = -2\sum_{n=1}^{\infty}\left[\frac{\cos(n\theta)}{n^3}+\theta\frac{\sin(n\theta)}{n^2}\right]S_1-\tfrac{1}{48}\theta^2(6\pi^2-8\pi\theta+3\theta^2)+\tfrac{1}{180}\pi^4\ , \tag{A.8}$$

$$\mathrm{Ls}_5^{(1)}(\theta) = \tfrac{3}{4}\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^3}\left(\frac{6}{n^2}-\frac{8}{n}S_1+4\left[S_1^2-S_2\right]-(\pi^2-\theta^2)\right)$$
$$+3\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}\left(\tfrac{1}{4}\theta(\pi-\theta)^2-\tfrac{1}{2}\frac{(2\pi-\theta)}{n^2}+(\pi-\theta)\frac{S_1}{n}+\theta\left[S_1^2-S_2\right]\right)+\tfrac{9}{2}\zeta_2\zeta_3-\tfrac{9}{2}\zeta_5\ , \tag{A.9}$$

$$\mathrm{Ls}_5^{(2)}(\theta) = 2\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}S_1\left(\frac{2}{n^2}-\theta^2\right)-4\theta\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^3}S_1-\frac{\theta^3}{120}\left(6\theta^2-15\pi\theta+10\pi^2\right), \tag{A.10}$$

where we have introduced a short notation $S_a \equiv S_a(n-1)$, and we used the general expression for the Stirling number of the first kind in terms of the harmonic sums [21]

$$\begin{aligned}
&\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{0n}\ , \quad \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!\ , \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!S_1\ ,\\
&\begin{bmatrix} n \\ 3 \end{bmatrix} = \tfrac{1}{2}(n-1)!\left[S_1^2-S_2\right]\ , \quad \begin{bmatrix} n \\ 4 \end{bmatrix} = \tfrac{1}{3!}(n-1)!\left[S_1^3-3S_1S_2+2S_3\right]\ ,\\
&\begin{bmatrix} n \\ 5 \end{bmatrix} = \tfrac{1}{4!}(n-1)!\left[S_1^4+8S_1S_3+3S_2^2-6S_4-6S_1^2S_2\right]\ ,\\
&\begin{bmatrix} n \\ 6 \end{bmatrix} = \tfrac{1}{5!}(n-1)!\left[S_1^5-30S_1S_4-20S_2S_3+20S_1^2S_3-10S_1^3S_2+15S_1S_2^2+24S_5\right]\ .
\end{aligned} \tag{A.11}$$

Using the results of paper [8] (see Eqs. (A.9), (A.10), (A.14) in [8]) we get from (A.5)-(A.9) the following relations:

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{3}n\right)}{n^2}S_1 = \tfrac{1}{54}\pi\zeta_2\ , \tag{A.12}$$

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{3}n\right)}{n^3}S_1 = -\tfrac{23}{216}\zeta_4\ , \tag{A.13}$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{3}n\right)}{n^2}\left[S_1^2-S_2\right] = 2\mathrm{Cl}_4\left(\tfrac{\pi}{3}\right)-\tfrac{2}{3}\zeta_2\mathrm{Cl}_2\left(\tfrac{\pi}{3}\right)-\tfrac{2}{9}\pi\zeta_3\ , \tag{A.14}$$

$$3\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{3}n\right)}{n^3}\left[S_1^2-S_2-2\frac{S_1}{n}\right]+2\pi\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{3}n\right)}{n^3}S_1 = 2\pi\mathrm{Cl}_4\left(\tfrac{\pi}{3}\right)-\tfrac{4}{3}\zeta_2\zeta_3-\tfrac{7}{3}\zeta_5\ , \quad \text{(A.15)}$$

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{2}n\right)}{n^3}S_1 = -\tfrac{181}{256}\zeta_4+\tfrac{5}{192}\ln^4 2-\tfrac{5}{32}\zeta_2\ln^2 2+\tfrac{35}{64}\zeta_3\ln 2+\tfrac{5}{8}\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ , \quad \text{(A.16)}$$

$$3\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi}{2}n\right)}{n^3}\left[S_1^2-S_2-2\frac{S_1}{n}\right]+\tfrac{3}{2}\pi\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{2}n\right)}{n^3}S_1 = \tfrac{3}{2}\pi\mathrm{Cl}_4\left(\tfrac{\pi}{2}\right)-\tfrac{93}{1024}\zeta_5-\tfrac{315}{256}\zeta_2\zeta_3$$
$$-\tfrac{1}{16}\ln^5 2+\tfrac{5}{16}\zeta_2\ln^3 2-\tfrac{105}{128}\zeta_3\ln^2 2-\tfrac{15}{8}\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln 2-\tfrac{15}{8}\mathrm{Li}_5\left(\tfrac{1}{2}\right)\ . \quad \text{(A.17)}$$

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